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LETTER TO THE EDITOR

Solution of an oscillator multiplicative stochastic equation as an ordering problem

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Abstract. An exact solution, compact in form, for the evolution operator of a harmonic oscillator driven by an external chaotic electric field is obtained. It is shown that using the harmonic description of the Ornstein–Uhlenbeck stochastic process, the evolution operator can be calculated explicitly from a normal ordering problem of certain Bose creation and annihilation operators.

Physical problems with external noise and fluctuations lead very often to stochastic equations with external multiplicative Ornstein–Uhlenbeck stochastic processes. Usually these equations are very difficult to treat because of the non-white character of the fluctuations involved in the problem.

In this paper we give an exact solution and, what is very important, one that is compact in form, for a harmonic oscillator

$$H=\omega_0 a^+a+\mathscr{E}^*(t)\,\mathrm{e}^{\mathrm{i}\omega t}a+\mathscr{E}(t)\,\mathrm{e}^{-\mathrm{i}\omega t}a^++\varepsilon\mathscr{E}^*(t)\mathscr{E}(t) \tag{1}$$

driven by an external near-resonant Gaussian chaotic electric field described by two complex amplitudes $\mathscr{E}(t)$ and $\mathscr{E}^*(t)$ (ε is a coupling constant singled out only for convenience). We obtain a closed form solution for the evolution operator of the harmonic oscillator, averaged statistically over the Ornstein–Uhlenbeck fluctuations of the external electric field defined by the following two-point correlation function:

$$\langle\mathscr{E}(t)\mathscr{E}^*(t')\rangle=(\Gamma/\tau_c)\exp(-|t-t'|/\tau_c). \tag{2}$$

For such a stochastic problem different techniques have been developed based on cumulant expansions (Fox 1979) or path integration methods (Kuś *et al* 1983). All these methods lead in general to quite complicated and involved calculations.

Following the harmonic description of the Ornstein–Uhlenbeck stochastic process (Wódkiewicz and Zubairy 1983), we can reduce the solution of the chaotic oscillator to an ordering problem of certain proper boson creation and annihilation operators. Writing the evolution operator in the form $\exp(-\mathrm{i}\omega t a^+a)U(t)$, we obtain for the Hamiltonian given by (1) the following operator-valued stochastic equation:

$$\mathrm{i}\dot{U}=(\Delta a^+a+\mathscr{E}^*a+\mathscr{E}a^++\varepsilon\mathscr{E}^*\mathscr{E})U \tag{3}$$

where $\Delta=\omega_0-\omega$.

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The complex chaotic electric field can be decomposed into two statistically independent stochastic processes x_1 and x_2: $\mathscr{E} = x_1 + \mathrm{i}x_2$. The real Ornstein–Uhlenbeck stochastic processes x_i can be described by Langevin equations with white-noise additive driving terms F_i:

$$\dot{x}_i = -\tau_c^{-1} x_i + F_i, \qquad i = 1, 2, \tag{4}$$

with

$$\langle F_i(t) F_j(t') \rangle = \delta_{ij} (2\Gamma/\tau_c^2) \delta(t - t').$$

The stochastic expectation value of the evolution operator U can be calculated from the following auxiliary quantity:

$$g(t, \xi_1, \xi_2) = \exp(\xi_1^2 \Gamma / 2\tau_c) \exp(\xi_2^2 \Gamma / 2\tau_c) \langle \exp(\mathrm{i}\xi_1 x_1(t)) \exp(\mathrm{i}\xi_2 x_2(t)) U(t) \rangle \tag{5}$$

by simply putting parameters ξ_1 and ξ_2 equal to zero:

$$\langle U(t) \rangle = g(t, \xi_1, \xi_2)|_{\xi_1 = \xi_2 = 0}. \tag{6}$$

Now an exact equation of motion for g can be derived differentiating (5) with respect to time. With the help of (1) and (4) we transform multiplications by the noise $x_i(t)$ into $\partial/\partial\xi_i$ derivatives, and according to the well known standard methods (Doob 1967) average over the white noise F_i, i.e. the only stochastic element left in our equation. As a result we obtain the equation

$$\dot{g} = (A^{\mathrm{T}} B A + b) g \tag{7}$$

where the operator-valued vector A is defined as follows:

$$A = (a, \partial/\partial\xi_1, \partial/\partial\xi_2; a^+, \xi_1, \xi_2) \tag{8}$$

and $b = -\mathrm{i}\Delta - \tau_c^{-1} - 4\mathrm{i}\Gamma\varepsilon/\tau_c$. The 6×6 matrix $B = \left|\begin{smallmatrix}\alpha & \gamma \\ \gamma^{\mathrm{T}} & \beta\end{smallmatrix}\right|$ is given by the following definitions:

$$\alpha = \begin{vmatrix} 0 & -\frac{1}{2} & \frac{1}{2}\mathrm{i} \\ -\frac{1}{2} & \mathrm{i}\varepsilon & 0 \\ \frac{1}{2}\mathrm{i} & 0 & \mathrm{i}\varepsilon \end{vmatrix}, \qquad \beta = \frac{\Gamma}{2\tau_c} \begin{vmatrix} 0 & 1 & \mathrm{i} \\ 1 & (\mathrm{i}\Gamma/\tau_c)\varepsilon & 0 \\ \mathrm{i} & 0 & (\mathrm{i}\Gamma/\tau_c)\varepsilon \end{vmatrix},$$

$$\gamma = \tfrac{1}{2} \begin{vmatrix} -\mathrm{i}\Delta & -1 & -\mathrm{i} \\ \Gamma/\tau_c & -1/\tau_c - (2\mathrm{i}\Gamma/\tau_c)\varepsilon & 0 \\ -\mathrm{i}\Gamma/\tau_c & 0 & -1/\tau_c - (2\mathrm{i}\Gamma/\tau_c)\varepsilon \end{vmatrix}. \tag{9}$$

From the definition (5) we check that (7) has the following initial condition:

$$g(t, \xi_1, \xi_2)|_{t=0} = I. \tag{10}$$

If we introduce the operators $a_i = \partial/\partial\xi_i$ and $a_i^+ = \xi_i$, we check that $[a_i, a_j^+] = \delta_{ij}$, i.e. boson commutation relations are satisfied leading to

$$[A^{\mathrm{T}}, A] = \begin{vmatrix} 0 & I \\ -I & 0 \end{vmatrix}. \tag{11}$$

This commutator of the operator-valued components of the vector A has an obvious symplectic structure.

From the initial condition (9) we conclude that operators a_i acting on the right give zero and operators a_i^+ acting on the left give zero due to the property (6). This means that the stochastic average of the evolution operator is simply equal to the vacuum expectation value (vacuum with respect to boson operator a_i only) of g given by (7), i.e.

$$\langle U(t)\rangle = e^{bt}\langle 0|\exp(A^{\mathrm{T}}BAt)|0\rangle. \qquad (12)$$

The calculation of this vacuum expectation value can be easily achieved by normally ordering all the boson operators entering in the expression (12). This ordering can be done exactly due to the bilinear structure on the creation and annihilation operators of the exponent in (12) (Berezin 1966, Wilcox 1967, Agrawal and Mehta 1977). As a result we obtain

$$\langle U(t)\rangle = f(t)\colon \exp(A_0^{\mathrm{T}}BT(t)A_0) \qquad (13)$$

where

$$A_0 = (a, 0, 0; a^+, 0, 0), \qquad (14)$$

$$T(t) = [\sinh(B_s t)/B_s]F^{-1}(t), \qquad (15a)$$

$$F(t) = \cosh(B_s t) + \begin{vmatrix} -I & 0 \\ 0 & I \end{vmatrix} \cdot \sinh(B_s t), \qquad (15b)$$

$$f(t) = e^{bt}(\det F(t))^{-1/2}, \qquad (15c)$$

with the notation $B_s = \begin{vmatrix} 0 & I \\ -I & 0 \end{vmatrix} \cdot B$.

The closed-form solution given by (13) is the main result of this paper.

This formula can be simplified further by performing a symplectic transformation S, i.e. a transformation leaving the relation (11) unchanged and diagonalising BT, i.e.

$$S^{\mathrm{T}}BTS = \begin{vmatrix} 0 & \lambda \\ \lambda & 0 \end{vmatrix} \qquad (16)$$

where λ is a diagonal matrix. As a result of such a transformation we can write

$$\langle U(t)\rangle = f(t)\colon \exp(g(t)a^+a) \colon = f(t)\exp(\ln|g(t)+1|a^+a), \qquad (17)$$

where we have used the known relation $\exp(\delta a^+a) =\colon \exp[(e^{\delta}-1)a^+a]\colon$ (Wilcox 1967) and the function $g(t)$ is obtained as a result of the symplectic transformation (16).

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